AIAA Guidance, Navigation, and Control Conference
10 - 13 August 2009, Chicago, Illinois

AIAA 2009-6222

# Radar Tracking Optimization for Ballistic Target Scenarios

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## Abstract

The problem of radar resource allocation in ballistic target scenarios is addressed. We search for robust solutions that take into account the possibility of new targets unknown to the process controller at the early phase of the tracking period. Two different optimization approaches to dynamic resources allocation are proposed and tested, showing significantly different results in terms of performance and robustness.

## 1. Introduction

A Ballistic Missile Defense (BMD) system uses its sensors for various missions such as detection and acquisition, tracking, fire control etc.

Missions "compete" over the same finite stockpile of sensor resources and have to be performed within certain time intervals. Mission performance level depends on the amount of sensor resources allocated to it and therefore can be optimized by specific allocation. In general, optimality criteria of sensor resource allocation may be scenario dependent.

An interim problem of the general resource optimization problem is radar tracking beam allocation. Classically, the objective of the sensor allocation process in tracking has been to maximize the information state (minimize the uncertainty in the tracking error) of all relevant targets using a given amount of radar resources. The first efforts to solve this problem took place in the sixties and early seventies of the last century. Athans and Schweppe [Ref 1] used the Kalman-Bucy theory to formulate an optimization problem for a single target where the states of the problem are the elements of the target's covariance matrix. The problem was constrained by the total

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amount of energy used and by the maximal peak power. They employed the Maximum Principle to derive the optimal on-off solution to the problem. A simpler model had been used by Schweppe and Gray [Ref. 2] where the special case of a one dimensional problem was considered under simplifying assumptions for which some basic structures of the optimal allocation have been revealed for constant velocity and constant acceleration targets.

The next profound effort was the work of Heffes and Horing [Ref. 3]. They considered the general case of multiple targets with different trajectories and proposed an iterative algorithm to solve the two-point boundary-value problem (TPBVP) derived from the Maximum Principle. The allocation of each and every pulse in open loop is done in a repeatable fashion.

Since the last task is time consuming, we can fix the allocation decisions over finite time intervals. Obviously the solution will not be optimal but it approximates the optimal allocation as these time intervals get smaller. Ref. 4 uses this approach to investigate the properties of optimal allocation strategies in tracking ballistic targets. The underlying assumption of Ref.4 is that the number of targets is fixed and is known to the allocation controller. The main purpose of the present work is to alleviate this assumption.

In determining radar resource allocation for “uncertain” scenarios, we search for robust solutions that take into account the possibility of new targets, unknown to the process controller at the early phase of the tracking period. We may want to sacrifice the optimality of the known set of targets in order to get better results with future targets that may show up.

In the present work, two approaches are compared to handle this problem. In the first approach we continue the line of Ref. 1-4 in minimizing the covariance of the known targets using maximal available resources. In the second and novel approach, we minimize future resources in order to save them for new targets.

The paper is organized as follows: the next section, together with the appendix, formulates the mathematical modeling of the problem. Section 3 introduces and explains the two allocation approaches. Results are given in Section 4 for a representative scenario, and a comparison between the two alternatives is performed. The last section concludes the paper.

## 2. Problem Formulation and Modeling

In general, the tracking problem can be modeled by the following set of equations (Ref. 4)

$$\begin{aligned} &x_j(i+1) = f_j(x_j(i), u_j(i)) \\ &i = 1..N \quad j = 1..n \end{aligned} \tag{1}$$

where $u_j$ is a binary number indicating which target is pulsed at time step i, $x_j$ is the predicated error covariance matrix of target j at a certain future (discrete) time N´, and N is the number of (tracking) time steps (thus $N \le N'$). There are n ballistic targets not necessarily identical.

By using a linearized Kalman Filter (KF), where the linearization is around a nominal ballistic trajectory, the set of equations for radar tracking a ballistic target can be further written as (Ref. 3):

$$\begin{aligned} &x_j(i+1) = x_j(i) - u_j(i) G_j(x_j(i), i) \\ &i = 1..N \quad j = 1..n \end{aligned} \tag{2}$$

where Gj represents the improvement in the prediction accuracy resulting from the assignment of the radar at the i$^{th}$ step to the j$^{th}$ target. For the complete derivation of $G_j$, see Appendix A.

In order to study the nature of the optimal strategies by direct optimization, and to keep the dimension of the problem small, we will search for approximate solutions (see Ref. 4 for more details).

To this end, we subdivide the tracking period into a relatively small number of segments L, where the control policy over each time segment is fixed. Let $p_{l,j}$, represents the fraction of the radar occupation time allocated to the j$^{th}$ target over the time interval $[t_l, t_{l+1}]$. In fact $p_{l,j}$ represents all combinations of fix number of pulses allocated the target j during that interval. Moreover, we will use the following approximation to Eq. (2):

$$
\begin{aligned}
&X_j(i+1) = X_j(i) - p_j(i) G_j(X_j(i), i) \\
&p_j(i) = p_{j,l} \quad for \quad i \cdot dt \in [t_l, t_{l+1}] \quad l = 0,1,\ldots L-1
\end{aligned}
\tag{3}
$$

where dt is the time step of the KF.

Notice again that the exact modeling for the covariance propagation should use

$$u_j(i) = \begin{cases} 1 & if \quad targets \quad is \quad pulsed \\ 0 & otherwise \end{cases}$$

for each KF step. Here we propose using a fixed $0 \le p_j(i) \le 1$ in the covariance propagation equation over the time interval $[t_l, t_{l+1}]$. In Ref.4 this approximation was numerically tested and validated.

Before we go on, we will justify the need for approximation. Suppose that we have a tracking period of 50 sec with the KF working in steps equal to the radar dwell time of, say, 100 milliseconds. We then have possible 500 pulses to allocate to all targets. Even if we know the number of pulses needed per target (which we don't), say it is 20, we still have $\binom{500}{20}$ possibilities to check for!

On the other hand, by subdividing the tracking period into 10 segments of 5 seconds each, we get only 10 unknowns (for each target) to deal with and optimize. Even with realistically many targets, it is a tractable non-linear programming (NLP) problem that can be solved in real time.

Thus, in our formulation, we seek for the best $p_j(i)$ in order to minimize a certain cost function. The next section considers two such functions.

## 3. Two Approaches for Robust Allocation

In realistic cases, we require allocation approaches that use the maximal available tracking resources at all times, i.e. there is no point in saving effort.
The following are two optional approaches:

**Approach 1**: Do the best you can in terms of accuracy improvement for the present set of targets, assuming the current situation persists. Deal with changes by adaptation.

**Approach 2**: Reduce the total future resources required for current job completion as much as possible. Lower the demand on future resources to enable allocating them to new targets should they appear.

An algorithm which accomplishes the first approach is:

**Algorithm 1**:

Repeat Calculate the optimal tracking resource allocation that:

a. Uses all available resources.

b. Minimizes the in-view targets combined prediction accuracy:

$$J = \sum_{j=1}^{n} w_j \cdot f_0(x_j(N_j')) \tag{5}$$

where $x_j$ error covariance at a future time $N'_j$, $f_0$ is a function of $x_j$ representing the variance in the predicted position error, and $w_j$ is a weighting factor for the $j^{th}$ target (some may be more important than others).

c. Execute the optimal allocation until end of track period or until a new target appears.

The main advantages of this approach are as follows:

a. In a known scenario it achieves the best possible combined accuracy.

b. The approach is relatively simple.

The main shortcomings of this approach are as follows:

a. React to new targets instead of planning ahead.

b. New targets may force us to abandon the previous optimal allocation. Disabling to complete this allocation may result in resource wastage (due to the bang-zero-bang nature of optimal tracking profiles [4])

An algorithm which accomplishes the second approach is:

**Algorithm 2:**

Partition tracking time into suitably small time steps, start at the first time step and repeat:

a. Calculate the tracking resource allocation that achieves the <u>required</u> accuracy $f_0((x_j(N_j'))$ for all in-view-targets using all available resources at the current time step and minimal total resources in future time steps.

b. Execute the allocation for the current time step only.

c. Move to the next time step.

The main advantages of this approach are as follows:

a. In a known scenario it achieves the required accuracy whenever possible.

b. It plans ahead for possible new targets.

The main shortcomings of this approach are as follows:

a. It does not acknowledge the value of improving the performance over the required accuracy

b. It is computationally complex

## 4. Representative Results

Assume a ballistic missile scenario with four representative targets launched at

times{0, 80, 140, 150} sec from the same site and follow the same trajectories to the defended asset. The targets are detected at a certain range from the radar. We predict to a future point where an altitude condition is met (different $N'_j$). We also terminate the tracking for each target separately, thus the tracking window is being shifted accordingly. The width of the tracking window for each target is 120 sec. The maximal resources allocated for tracking are $\sum_{j=1}^{n} p_j = 0.5$ in one case (Case a), and $\sum_{j=1}^{n} p_j = 0.35$ in the more stressing second case (Case b). It is assumed that the required accuracy is 2500 m for each target.

**Case a**:

The first approach's optimal allocation and time histories of the prediction error are shown in Fig. 1 and Fig. 2. The optimal results at the end of tracking are {1425m; 1472 m; 1797m; 1638m} exceeding the requirements.

The second approach's optimal allocation and time histories of the prediction error are shown in Fig. 3 and Fig. 4. The optimal results at the end of tracking phase are {2500m; 2500 m; 1644m; 1393m} meeting the requirements for the first two targets and exceeding them for the others. As might be expected, the statistical sum J (Eq. (5)) is lower in the first approach being its direct cost function. This optimality is obtained by improving the first targets' accuracy beyond the requirements.

**Case b:**

This is a more stressing case and the results of the first approach are {2355 m; 2256 m; 2360m; 2549m} not meeting the requirements for the last target. The second approach however obtains {2500 m; 2500 m; 2500m; 1990m}, thus although still less optimal in the sense of J, it does meet the requirements for all targets.

## 5. Conclusions

Two different approaches to dynamic radar resources allocation in tracking ballistic targets were tested, showing significantly different results in terms of performance and robustness.

Approach 1 is greedy and tries to reduce the error (beyond the requirements) for all targets in view. Approach 2 does not use resources for improving performance beyond requirements, as long as they can still be used for non-convergent targets. As a result, the approach is more robust for unpredicted targets. We are facing a classical dilemma of performance vs. robustness

## References

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## Appendix A: Modeling

For simplicity we omit the target's index j.

For the target's dynamics we use a flat earth 2-D scenario governed by the following dynamic equations:

$$\begin{aligned}
\frac{dx}{dt} &= f_1(x,h,v,\gamma) = v\cdot\cos(\gamma)\\
\frac{dh}{dt} &= f_2(x,h,v,\gamma) = v\cdot\sin(\gamma)\\
\frac{dv}{dt} &= f_3(x,h,v,\gamma) = -\frac{1}{2}\rho(h)v^2\frac{C_DS}{m} - g\cdot\sin(\gamma) + w_1;\\
\frac{d\gamma}{dt} &= f_4(x,h,v,\gamma) = -\cos(\gamma)\cdot\frac{g}{v} + \frac{w_2}{v}
\end{aligned} \tag{A1}$$

Where:

x – range

h – altitude

v - velocity

$\gamma$ – dive angle

m - mass

$C_D$ – drag coefficient

S – reference area

$\rho$ – density

g – gravity

$w_1, w_2$ – drag and lift disturbances

Let the radar measures slant rage and LOS angles to target. Thus:

$$
\begin{aligned}
y_k &= \underline{g}_k + \underline{v}_k \quad , \quad \underline{v}_k \sim N(0,R_k)\\
\underline{g}_k &= \left[\Psi_k \quad \Theta_k \quad r_k\right]^T\\
\begin{bmatrix}\Psi_k\\ \Theta_k\\ r_k\end{bmatrix} &= \left\{\begin{array}{c}
\arctan\left(\dfrac{y_k - y_{r,k}}{x_k - x_{r,k}}\right)\\
\arcsin\left(\dfrac{z_{r,k} - z_k}{r_k}\right)\\
\sqrt{(x_{r,k}-x_k)^2 + (y_{r,k}-y_k)^2 + (z_{r,k}-z_k)^2}
\end{array}\right\}
\end{aligned}
\tag{A2}
$$

Where $(\ )_k$ denotes values at discrete time k and $(\ )_{r,k}$ denotes values of radar at this time, and where x,y,z are Cartesian coordinates; $\Psi$ – azimuth ; $\Theta$ – elevation; r – range. Since we are dealing with a 2-D scenario, we will ignore the azimuth in the sequel.

To obtain the error propagation equation, we linearize around a nominal trajectory and we get an expression for the covariance matrix P. When measurements are taken, we get the following improvement in the current covariance matrix:

$$
\begin{aligned}
&P = M - MC^T[CMC^T + R]^{-1}CM \\
&M = APA^T + LQL^T \\
&A = \frac{\partial f_i}{\partial x_j}; \qquad C = \frac{\partial g_i}{\partial x_j} \\
&L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{v} \end{bmatrix}^T; \quad Q = \begin{bmatrix} \sigma_{w_1}^2 & 0 \\ 0 & \sigma_{w_2}^2 \end{bmatrix}; \quad R = \begin{bmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}
\end{aligned}
\tag{A3}
$$

In the prediction step (no measurements) we use

$$
\begin{aligned}
&P = M \\
&M = APA^T + LQL^T
\end{aligned}
\tag{A4}
$$

Thus when no measurement are taken, the prediction values are not changed. The value of G in Eq. (2) can be shown, after some calculations, to have the following form (Ref. 3):

$$
G(x(i),i) = x(i)[A^{N'-i}]^{-1}C^T\left[C[A^{N'-i}]^{-1}x(i)[A^{N'-i}]^{-T}C^T + R\right]^{-1}C[A^{N'-i}]^{-1}x(i) \equiv x(i)S(i)
\tag{A5}
$$

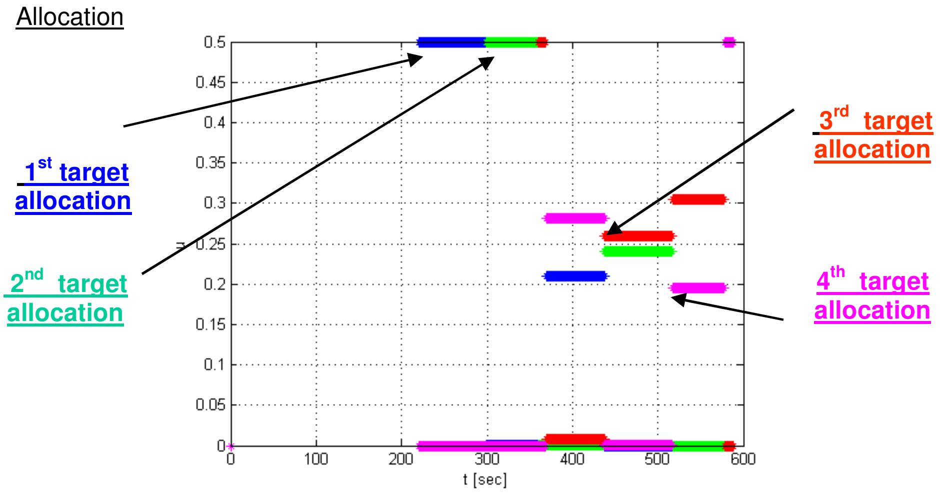

Fig 1: Case a, Approach 1 – allocation values

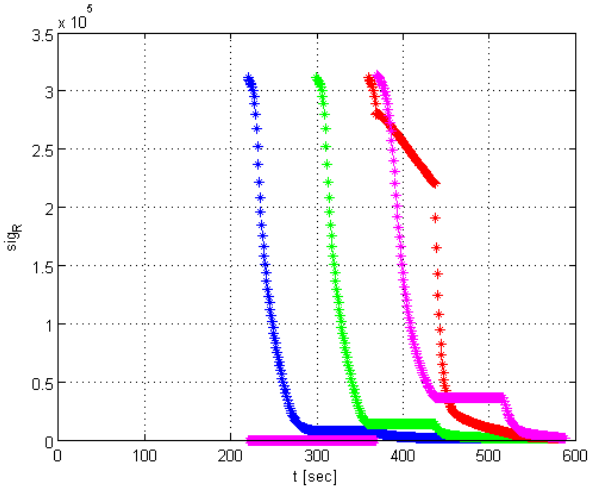

Fig 2: Case a, Approach 1: predicted accuracies

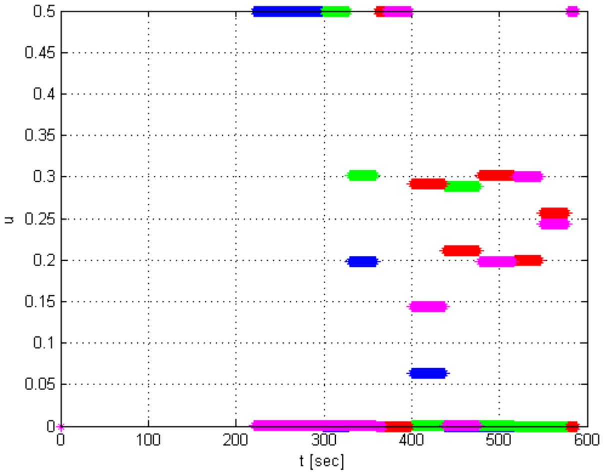

Fig 3: Case a, Approach 2 – allocation values

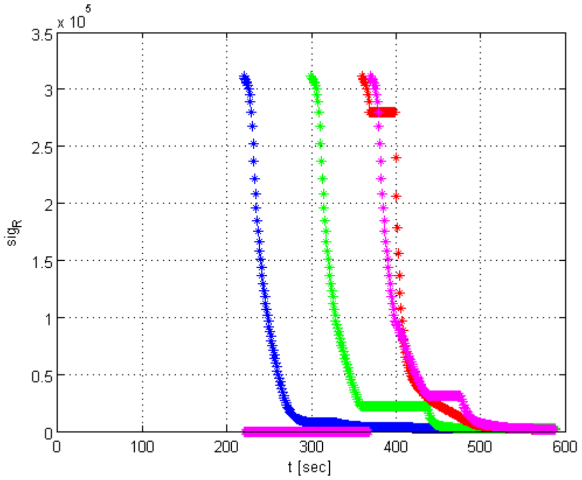

Fig 4: Case a, Approach 2 : predicted accuracies