

Applications of Analytic Expressions for Rocket Trajectory

Joseph Z. Ben-Asher,

Faculty of Aerospace Engineering
Technion - Israel Institute of Technology, 32000 Haifa, Israel

Dror Cohen, Michael Wetzler

WALES, Ltd. 11 Tuval St. Ramat Gan, 52522,
Tel: 972-3-7567200, Fax: 972-3-7510917, EMail: wales@walesltd.co.il

Abstract

Modeling and simulation tasks are important for the development of rocket systems and for analyzing various performance issues. In carrying out these tasks, the trajectories of rockets and projectiles are typically obtained via numerical integration of the equations of motion for atmospheric flight. These equations are taking into account gravity, thrust and drag forces. The purpose of this paper is to obtain and exploit approximate analytic expressions for the ballistic trajectory. In the reported work we utilized theoretical analysis performed in recent years by Chudinov for ballistic trajectories limited for objects thrown at low velocity and low Reynolds number, (e.g. baseball), and expanded it to deal with trajectories of rockets. We analyzed the validity of the formulas for rockets trajectory and for two applications: prediction of Ground Impact Point (GIP) and contact envelopes. Although the analytic expression applied in the examples assumed constant ratio between drag forces and the squared velocity, the application to realistic rocket model was quite successful.

1. Introduction

Most artillery and ballistic rocket studies require the generation and analysis of a large number of rocket trajectories. Currently, such studies and applications use numerical simulations to create and run these trajectories. Analytic expressions, even approximate ones, can be very useful and can serve at least three purposes: saving computation time, providing insight into the physical phenomena and enabling optimization algorithms.

In the past, many works considered analytic expressions for two degrees of freedom ballistic trajectory. The simplest solution is parabolic trajectory in vacuum. Timoshenko and Young [1] solve several exterior ballistics problems for constant ballistic coefficient e.g. exact solution for the case of drag proportional to the square of the velocity and approximation for more complex cases. Chudinov [2] proposed new results using a simple approximation of the drag influence on the ballistic trajectory.

Most of the examples demonstrating Chudinov's analytic approximations to ballistic trajectories are taken from sport [2]-[4]. The objective of this study is to highlight some recent results concerning analytic expressions of ballistic rocket trajectories, applying Chudinov's method.

The validation of Chudinov's method for rockets will be presented by comparing (more) accurate results with the analytic approximations. In the sequel we consider under the terminology of rockets: rockets in free flight, projectiles and mortars. A parametric study reveals when these approximations are valid. Further, two applications are given in order to demonstrate the usefulness of the analysis for rocket systems: Ground Impact Point (GIP) prediction and contact envelope and atmospheric keep-out zones.

2. Rocket Trajectory Equations

Consider the point mass 2 DOF dynamic equations for the motion of a rocket in post-boost flight. The basic motion parameters used are described in Figure 1:

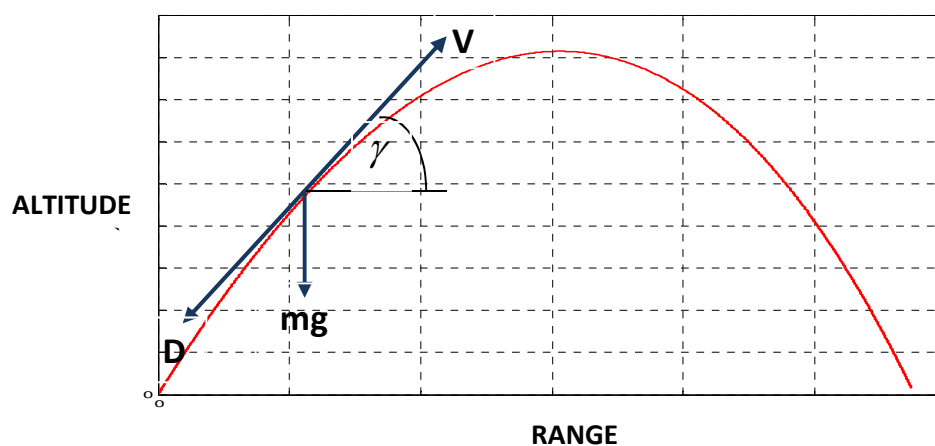


Figure 1. Point Mass trajectory Notation

Assuming flat non-rotating earth approximation, constant aerodynamic drag and constant air density, the equations of ballistic trajectory (no lift) become:

$$\begin{aligned}\dot{V} &= -k g V^2 - g \sin \gamma \\ \dot{\gamma} &= -\frac{g}{V} \cos \gamma \\ \dot{h} &= V \sin \gamma \\ \dot{x} &= V \cos \gamma\end{aligned}\tag{Eq. 1}$$

where: $k \equiv \frac{\rho S C_D}{2mg}$, x and h are the range and the altitude, respectively, in Cartesian coordinates, V is the velocity magnitude, γ is the flight-path angle, ρ is the air density, S is rocket cross section area, C_D is the drag coefficient and g is the gravity.

Dividing the first two equations we get:

$$\begin{aligned}\frac{dV}{d\gamma} &= \frac{k V^3 - V \sin \gamma}{\cos \gamma} \\ \frac{dV}{d\gamma} - V \tan \lambda &= \frac{k V^3}{\cos \gamma}\end{aligned}\tag{Eq. 2}$$

Hence:

$$\frac{d(V \cos \gamma)}{(V \cos \gamma)^3} = k \frac{d\gamma}{(\cos \gamma)^3}\tag{Eq. 3}$$

After integration (e.g. using Table of Integrals), we get the following relationship between flight path angle and the velocity (sometimes denoted as Hodograph)¹:

$$V^2 = \left[\left(\frac{V_0 \cos \gamma_0}{\cos \gamma} \right)^2 - k \cos^2 \gamma \left(\sec^2 \gamma \tan \gamma - \sec^2 \gamma_0 \tan \gamma_0 + \ln \left(\frac{\sec \gamma + \tan \gamma}{\sec \gamma_0 + \tan \gamma_0} \right) \right) \right]^{-1}\tag{Eq. 4}$$

Applying Chudinov's approximation, the following results are obtained for the rocket trajectory at **apogee**: velocity V_a , maximum height H , time to apogee t_a and range to apogee x_a .

¹ This is a former result by Timoshenko and Young, 1948

$$\begin{aligned}
V_a &= \frac{V_0 \cos \gamma_0}{\sqrt{1 + kV_0^2 [\sin \gamma_0 + \cos^2 \gamma_0 \cdot \ln(\tan(\gamma_0/2 + \pi/4))]}]} \\
H &= h - h_0 = \frac{V_0^2 \sin^2 \gamma_0}{g(2 + kV_0^2 \sin \gamma_0)} \\
t_a &= \frac{T - kHV_a}{2} \\
x_a &= x - x_0 = \sqrt{LH \cot \gamma_0}
\end{aligned}
\tag{Eq. 5}$$

The **terminal conditions**: total range L , final time T , flight path angle γ_f and terminal velocity V_f are approximated as well:

$$\begin{aligned}
T &= t - t_0 = 2\sqrt{\frac{2H}{g}} \\
L &= x - x_0 = V_a T \\
\gamma_f &= -\arctan\left[\frac{LH}{(L - x_a)^2}\right] \\
V_f &= V(\gamma_f)
\end{aligned}
\tag{Eq. 6}$$

Finally, after additional development, we get the trajectory Equation 7 which will apply in our examples detailed herein:

$$h(x) - h_0 = \frac{H(x - x_0) \cdot (L - (x - x_0))}{x_a^2 + (L - 2x_a)(x - x_0)}
\tag{Eq. 7}$$

For zero initial conditions ($x_0 = h_0 = 0$):

$$h(x) = \frac{Hx \cdot (L - x)}{x_a^2 + (L - 2x_a)x}
\tag{Eq. 8}$$

Note that for zero drag, $L = 2x_a$

3. The Contact Envelope of Rocket Trajectories in Midair

Our interest is to determine the envelope equation of rockets trajectories which are launched with the same initial velocity V_0 but different angels γ_0 . The equation of envelope proposed by Chudinov [2] is:

$$y(x) = \frac{H_{\max}(L_{\max}^2 - x^2)}{L_{\max}^2 - ax^2}
\tag{Eq. 9}$$

where H_{\max} is the maximum altitude reached for the initial conditions of $V_0, \gamma_0 = 90^\circ$ and given by:

$$H_{\max} = \frac{1}{2gk} \ln(1 + kV_0^2) \quad \text{Eq. 10}$$

L_{\max} is the maximum range reached when the rocket is launched at an optimal initial angle α given by the following equation:

$$\tan^2 \alpha + \frac{p \sin \alpha}{4 + 4p \sin \alpha} = \frac{1 + p\lambda}{1 + p(\sin \alpha + \lambda \cos^2 \alpha)} \quad \text{Eq. 11}$$

$$\lambda = \ln \left(\tan \left(\frac{\alpha}{2} + \frac{\pi}{4} \right) \right)$$

4. Analytic Trajectory Expression Validation

Chudinov's analytic trajectory expression was validated for rockets by comparing **2DOF numerical simulation** results to the analytical approximations. In the simulation a realistic drag coefficient profile was used as depicted in Figure 2 and the density was assumed exponential with height. Note that in the simulation k is now a non-linear function of the altitude and speed, whereas under Chudinov's approach it is assumed constant.

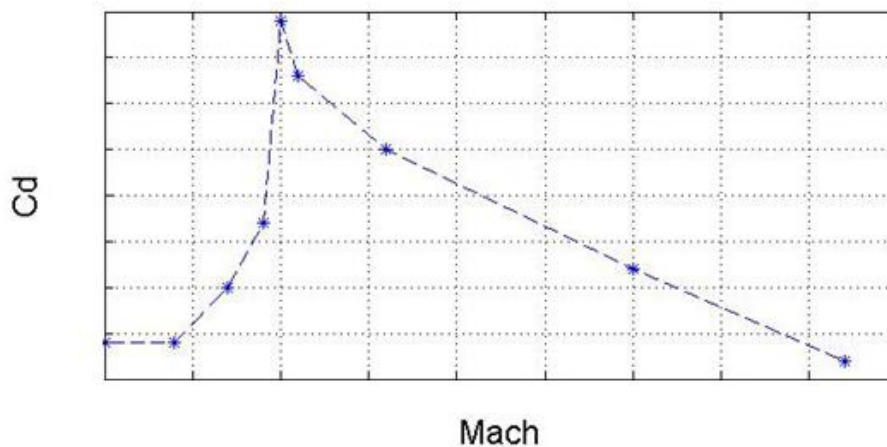


Figure 2. Realistic Rocket Drag Coefficient

Recalling the equation for k we point out that it is actually the reciprocal of the terminal velocity squared (defined as the steady state solution of the velocity equation in vertical throw):

$$k = \frac{\rho S C_D}{2mg} = \frac{1}{(V_{term})^2} \quad \text{Eq. 12}$$

In order to validate the Chudinov approximation with respect to rocket trajectory, the parameter $p = kV_0^2 = \left(\frac{V_0^2}{V_{term}^2}\right)$ is defined, which represents how much energy was dissipated due to drag. Two cases are checked: $p < 1$ where the gravity force is the dominant one and $p > 1$ where the drag force has more influence on the trajectory.

In Figure 3, simulation results are compared to the analytic approximation for $p < 1$. The comparison indicates acceptable accuracy for the approximation in both the trajectories and the contact envelope.

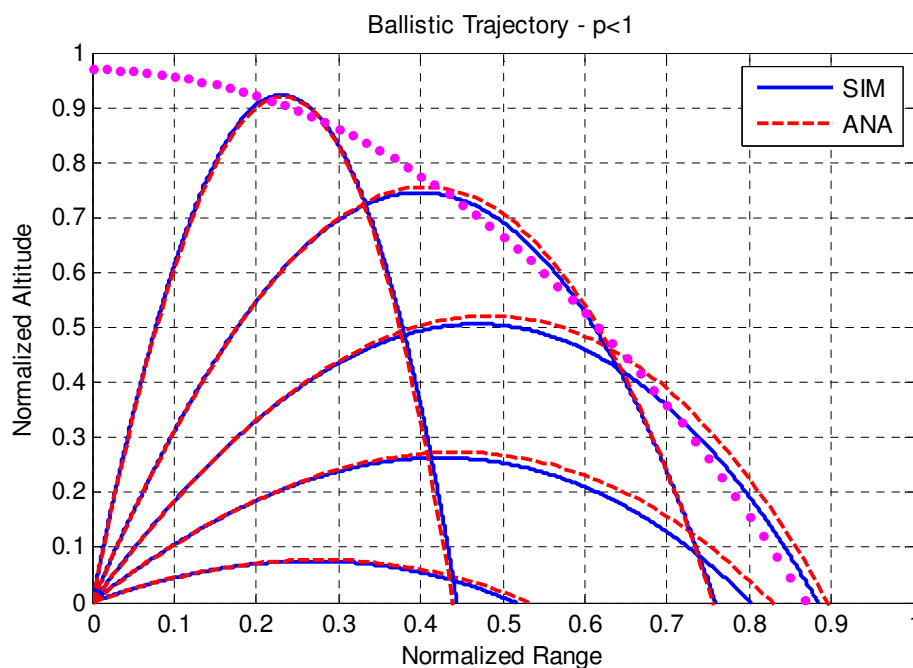


Figure 3. Simulation vs. Chudinov Approximation, $p < 1$

In Figure 4, results for $p > 1$ are depicted. Neither the trajectory profiles nor the contact envelop approximation are acceptable. Any attempt to tune the constant parameter k led to wrong results and no fit was achieved.

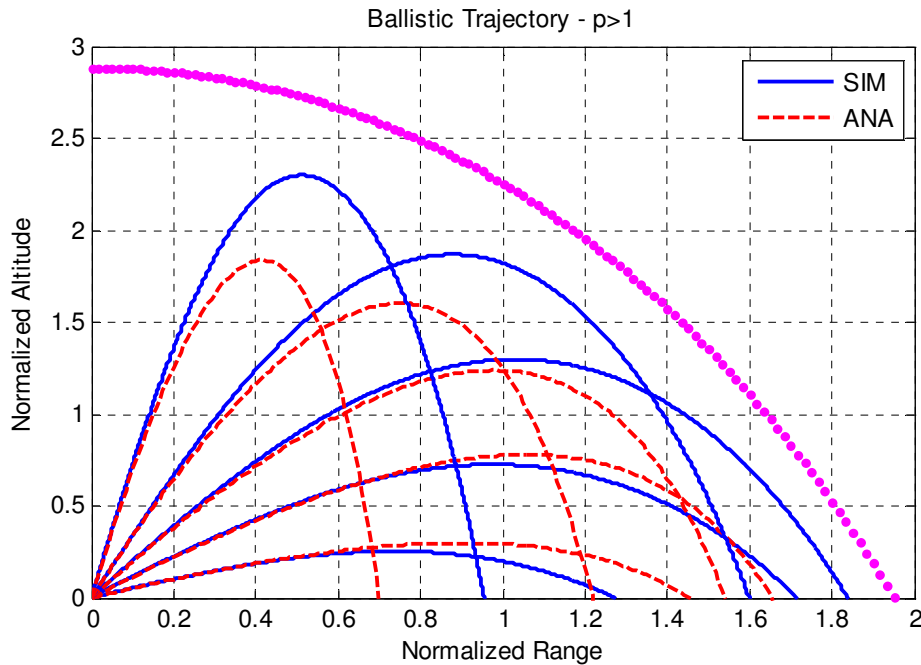


Figure 4. Simulation vs. Chudinov Approximation, $p > 1$

5. Application of Chudinov Formulas for $p > 1$

It turns out that in the case of $p > 1$, a priori knowledge of some parameters of a generic rocket trajectory can dramatically improve the results calculated by Chudinov's approximation.

Assuming $x_0 = h_0 = 0$, and using the triplet $\{H, x_a, L\}$ (either taken from flight tests or from simulations), three points on the realistic trajectory are known, namely: the launch point $(0, 0)$, the apogee (x_a, H) and the ground impact point $(0, L)$. We apply Equation 9 to approximate (in fact - to interpolate) the flight profile. Figure 5 compares exact **3DOF simulation** results to the approximation. As an outcome of the comparison, the maximum error between the two curves is less than 1% for a typical mid-range rocket.

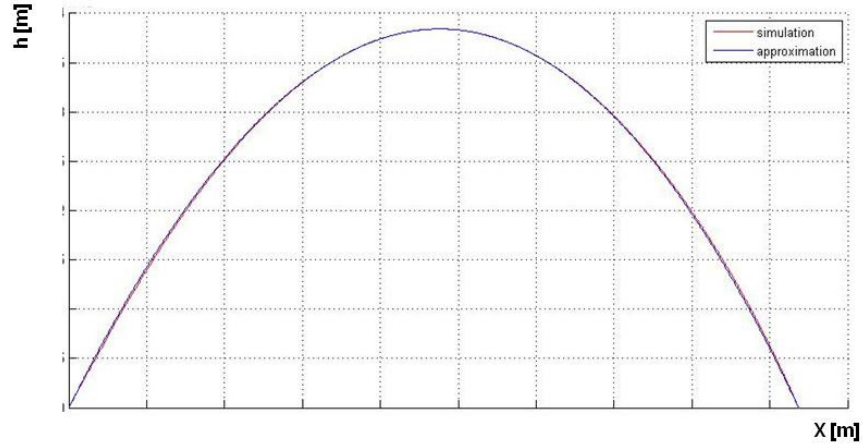


Figure 5. Rocket Trajectory – Simulation vs. Approximation Results

6. Ground Impact Prediction (GIP)

Consider a set of n noisy radar measurements of the altitude and the range of a rocket along the trajectory $\{x_i, h_i\}; i = 1:n$. For simplicity, we assume here that the radar polar measurements have been transformed to Cartesian. We also assume that it is known that the launch is from sea level. **Note that** these two assumptions can be easily relieved. An important task for the defense systems is to predict the GIP. To this end, we propose here to use Least Square Fitting (LSF) of the parameters $\{x_0, H, x_a, L\}$ in Equation 7. Given x_i the estimated values for altitudes become:

$$\hat{h}_i = f(x_i; x_0, H, x_a, L); \quad i = 1:n \quad \text{Eq. 13}$$

Where

$$f(x) = \frac{H(x - x_0) \cdot (L - (x - x_0))}{x_a^2 + (L - 2x_a)(x - x_0)} \quad \text{Eq. 14}$$

Applying LSF we would like to minimize the error between the measurement of $h(x)$ and its estimation by optimizing the four parameters:

$$\min_{\{x_0, H, x_a, L\}} S = \sum_{i=1}^n (\hat{h}_i - h_i)^2 \quad \text{Eq. 15}$$

Figure 6 depicts the simulated trajectory '-', to which the radar noisy measurements 'o' was added, and finally the trajectory estimated from the first measurements using LSF. As shown, the approximation provides a good estimation of the Ground Impact Point. It should be noted that the LSF method enables consistent estimation of the predicted error.

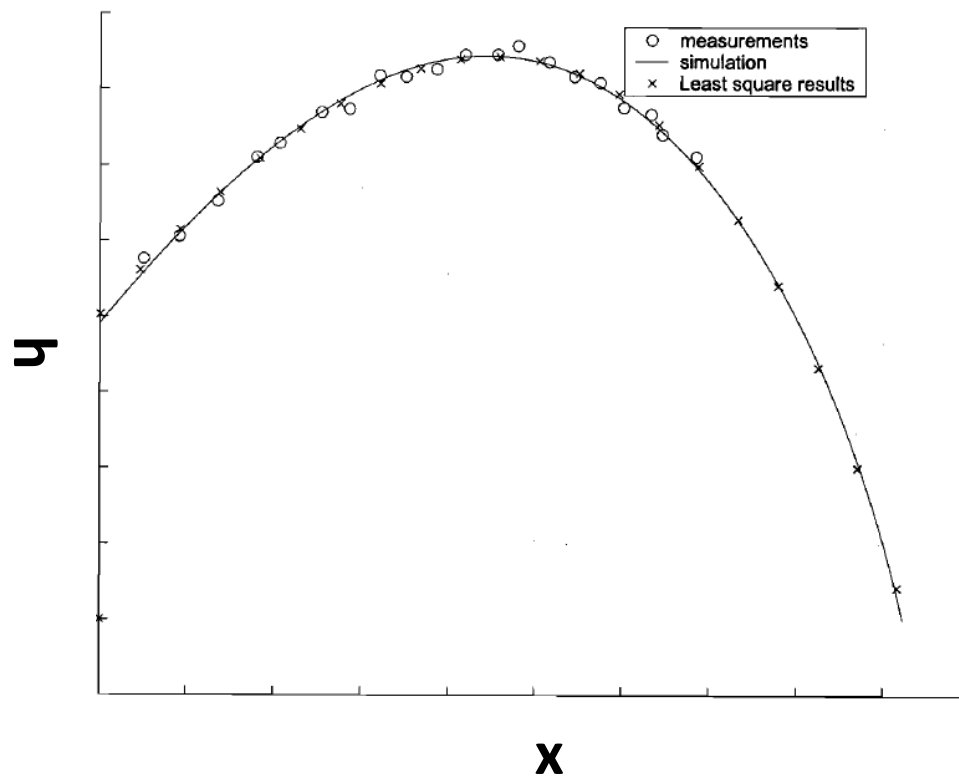


Figure 6. Rocket Trajectory and Radar Measurements

7. Contact Envelope

In our context, we would like to define a keep-out-zone for aircraft which is a restricted area to fly in from safety considerations. Flight is confirmed outside the contact envelope as will be demonstrated in this paragraph.

Here again, Chudinov tested these results for constant k (like in sport games). It is of great interest to test them for actual rockets. We are assuming $x_0 = h_0 = 0$, and that the triplet $\{H, x_a, L\}$ is known. Figure 8 shows the contact envelope calculated using Chudinov's formulas for rocket trajectories with realistic drag. As shown this

envelope approximation is quite good for keep-out-zone determination. However some safety margin should be added.

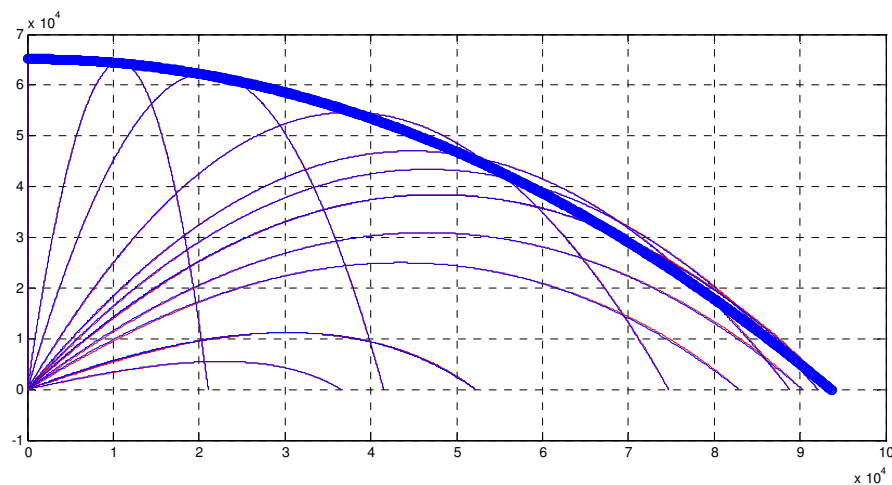


Figure 7. Realistic Drag Envelope.

8. Conclusion

Based on Chudinov's recent works, approximate analytic expressions for trajectories of rockets have been presented. The approximations were tested against detailed and accurate simulations. A parameter p has been identified to evaluate the results. Two important applications have been proposed for ballistic missile defense systems: predicting GIP and estimating contact envelopes.

To sum up: although the analytic expressions developed by Chudinov assuming a constant ratio between the drag forces and the squared velocity, the applications to realistic rocket models are quite promising.

9. References

- 1 Timoshenko and Young, Advanced Dynamics, McGraw –Hill Book Company, Inc., 1948
- 2 Peter S Chudinov, The Envelope of Projectile Trajectories in Midair, Proceedings of Fourth Okuner Readings, June 2004, St. Petersburg, Russia
- 3 Texier et all, On the size of sports fields, New Journal of Physics 16 (2014) 033039
- 4 Peter S Chudinov, Extension of application field of analytical formulas for the computation of projectile motion in midair, Revista Brasileira de Ensino de Física, v. 35, n. 1, 1310 (2013)